namibia university
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: Bachelor of Science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 07BAMS	**LEVEL:** 7
COURSE CODE: CAN702S	**COURSE NAME:** COMPLEX ANALYSIS
SESSION: NOVEMBER 2019	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

FIRST OPPORTUNITY EXAMINATION QUESTION PAPER	
EXAMINER	PROF. G. HEIMBECK
MODERATOR:	PROF. F. MASSAMBA

INSTRUCTIONS	
	1. Answer ALL the questions in the booklet provided.
	2. Show clearly all the steps used in the calculations.
	3. All written work must be done in blue or black ink and sketches must be done in pencil.

PERMISSIBLE MATERIALS

1. Non-programmable calculator without a cover.

THIS QUESTION PAPER CONSISTS OF 4 PAGES (Including this front page)

Question 1 [13 marks]

a) State the definition of a field. [4]

b) Outline the construction of the field of the complex numbers which has been carried out in class. Proofs are not required. [5]

c) Consider the relation $i = \sqrt{-1}$. Is this the definition of i? What is the meaning of this relation? Is $-i = \sqrt{-1}$ true? Explain. [4]

Question 2 [10 marks]

a) Complex conjugation is an involutory automorphism of the field $\mathbb{C}$ with fixed point field $\mathbb{R}$. Explain what this statement means. [4]

b) Let $f : \mathbb{C} \to \mathbb{C}$ be an involutory automorphism of $\mathbb{C}$ with fixed point field $\mathbb{R}$.

   i) Find $f(i)$. [3]

   ii) Show that f is equal to complex conjugation. [3]

Question 3 [13 marks]

a) If $z \in \mathbb{C}$, state the definition of $|z|$. Show that the absolute value of the field $\mathbb{C}$, which is used in complex analysis, is an extension of the absolute value of the field $\mathbb{R}$. [4]

b) Let $z, w \in \mathbb{C}$.

   i) Find $|z + w|^2 - |z|^2 - |w|^2$. [4]

   ii) Show that
$$|z + w|^2 = |z|^2 + |w|^2 \qquad \Longleftrightarrow \qquad \mathrm{Re}(z\bar{w}) = 0.$$
[5]

Question 4 [15 marks]

You are reminded of the definition of the circle group S:

$$S := \{z \in \mathbb{C} \mid |z| = 1\}.$$

a) Let $f : \mathbb{R}^+ \to S$ be defined by

$$f(\varphi) := \cos \varphi + (\sin \varphi) i.$$

Show that f is a surjective homomorphism and find the kernel of f. [9]

b) Let $\varphi \in \mathbb{R}$ and $k \in \mathbb{Z}$. Since f is a homomorphism, $f(\varphi)^k = f(k\varphi)$ is true. Write down this formula without using f. What is the name of this formula? [3]

c) Make the modulus-argument form of a non-zero complex number. [3]

Question 5 [19 marks]

Let $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) := \sum_{k=0}^{n} a_k z^k.$$

a) What is the degree of f? State the definition. [3]

b) Now assume that $n \in \mathbb{N}$ and $a_n \neq 0$. If $z \in \mathbb{C}$ such that $|z| \geq \max\{1, \frac{1}{|a_n|} \sum_{k=0}^{n-1} |a_k|\}$, show that

$$|f(z)| \geq \frac{1}{2} |a_n| |z|^n.$$

[8]

c) Prove that $\lim_{z \to \infty} f = \infty$. [8]

2

Question 6 [15 marks]

Let $\sum a_k z^k$ be a power series. The summation starts at 0. Let $\varepsilon > 0$ such that $N_\varepsilon(0)$ is contained in the set of convergence of $\sum a_k z^k$ and let $f : N_\varepsilon(0) \to \mathbb{C}$ be defined by

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

Assume that f is not constant.

a) Why does $m := \min\{k \in \mathbb{N}|\ a_k \neq 0\}$ exist? Explain! [2]

b) Show that

$$f(z) = a_0 + z^m \sum_{k=0}^{\infty} a_{k+m} z^k,$$

for all $z \in N_\varepsilon(0)$. [4]

c) Prove that 0 is an isolated point of $f^{-1}(0)$. [9]

Question 7 [15 marks]

a) State Goursat's lemma. This lemma is important for two reasons. State these reasons. [4]

b) Let $O \subset \mathbb{C}$ be open and $a \in O$. Let $f : O \to \mathbb{C}$ be a function which is holomorphic in $O - \{a\}$ and continuous at a. Let $\{a, b, c\}$ be a triangle such that the convex hull $\langle a, b, c \rangle$ of $\{a, b, c\}$ is contained in O. Prove that

$$\int_{[a,b]+[b,c]+[c,a]} f(\zeta)\, d\zeta = 0.$$

[8]

c) State Cauchy's integral formula for a disc. [3]

End of the question paper